Gamma Function

The gamma function is defined by

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx, \quad n > 0 \quad \text{---(1)}$$

This integral Converges uniformly w.r.t n. Therefore $\Gamma(n)$ is a Continuous function.

Integrating by parts, the equation (1), we have

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}\, dx$$

$$= \left[\frac{x^n e^{-x}}{-1} \right]_0^\infty - \int_0^\infty n x^{n-1} \cdot \frac{e^{-x}}{-1}\, dx$$

$$= \left[-x^n e^{-x} \right]_0^\infty + n \int_0^\infty x^{n-1} e^{-x}\, dx$$

$$= 0 + n\Gamma(n)$$

Thus $\Gamma(n+1) = n\Gamma(n)$

In Particular, putting $n = n, n-1, n-2 \cdots 1$ successively (n being a natural number) we have

$$\Gamma(n+1) = n\Gamma(n)$$

$$\Gamma(n) = (n-1)\,\Gamma(n-1)$$

$$\cdots\cdots\cdots$$

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$\Gamma(2) = 1\Gamma(1)$

Thus $\Gamma(n+1) = n(n-1)(n-2)\cdots 1\,\Gamma(1)$

But $\Gamma(1) = \int_0^\infty e^{-x}dx$

$= -\left[e^{-x}\right]_0^\infty = 1.$

Therefore, $\Gamma(n+1) = n(n-1)(n-2)\cdots 1$
where n is a natural number

$= n!$

If we substitute $y = \frac{x}{a}$ in (1)

$\Gamma(n) = \int_0^\infty (ay)^{n-1} e^{-ay} a\,dy$

$= a^n \int_0^\infty y^{n-1} e^{-ay} dy$

$$\therefore \boxed{\frac{\Gamma(n)}{a^n} = \int_0^\infty y^{n-1} e^{-ay} dy}$$

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Beta Function :-

The beta function is defined by

$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx\,; \quad m>0, n>0$

(I) To prove that $B(m,n) = B(n,m)$

Proof. We have

$B(n,m) = \int_0^1 x^{n-1}(1-x)^{m-1}dx$

We put $y = 1-x$ so that when $x=0$, $y=1$

$dy = -dx \Rightarrow dx = -dy$

$\therefore B(n,m) = \int_1^0 (1-y)^{n-1} y^{m-1}(-dy)$

$= -\int_1^0 y^{m-1}(1-y)^{n-1}dy$

$= \int_0^1 x^{m-1}(1-x)^{n-1}dx$

$= B(m,n)$

Proved

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(II) To prove that $\displaystyle\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = B(m,n)$

We have $\displaystyle\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

We put $x = \dfrac{y}{1-y}$

So that $1+x = 1+\dfrac{y}{1-y}$

$$= \frac{1-\cancel{y}+\cancel{y}}{1-y} = \frac{1}{1-y}$$

$\therefore\; dx = \dfrac{1}{(1-y)^2}\,dy$

Also $x=0 \Rightarrow y=0$ &

$x=\infty \Rightarrow y=1$

$$\therefore \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^1 \frac{y^{m-1}}{(1-y)^{m-1}}\cdot(1-y)^{m+n}\cdot\frac{dy}{(1-y)^2}$$

$$= \int_0^1 y^{m-1}(1-y)^{n-1}\,dy$$

$$= B(m,n)$$